$\cup$-shaped and takes its maximum value over the domain of allowed values of t at the endpoint $t = 1 - \sin\frac{1}{2}A$, corresponding to the isosceles triangle with $x = \frac{1}{2}(\pi - A) = B = C$.

If $\pi\tan\frac{1}{2}A - 1 > 0$ or $A > 2\tan^{-1}\frac{1}{\pi}$, the quadratic graph of $Q(t)$ is $\cap$-shaped, but the value of t giving its global maximum occurs at $t = \dfrac{\sin\frac{1}{2}A}{\pi\tan\frac{1}{2}A - 1}$ which (check!) is always greater than $1 - \sin\frac{1}{2}A$, the largest allowed value of t. So in this case also, there is an endpoint maximum corresponding to an isosceles triangle with apex angle A.

We now use a lovely old argument (which may be traced back to Simon Lhuilier (1750-1840), [1]) to deduce that, among all triangles, the largest value of Q is given by an equilateral triangle. Certainly, since the expression for Q is continuous in (A, B, C), it attains its global maximum on the compact region $0 \leqslant A, B, C \leqslant \pi$, $A + B + C = \pi$. And, starting with an arbitrary triangle with angles (A, B, C), the isosceles triangle with apex angle A and angles $\left(A, \frac{1}{2}(B + C), \frac{1}{2}(B + C)\right)$ has larger Q-value; repeating this argument shows that the isosceles triangle with angles $\left(\frac{1}{2}(B + C), \frac{1}{2}(A + \frac{1}{2}(B + C)), \frac{1}{2}(A + \frac{1}{2}(B + C))\right)$ has even larger Q-value. Iterating this argument generates a sequence of isosceles triangles with ever-increasing Q-values which always converge to the equilateral triangle, since the greatest difference between the angles of the nth isosceles triangle in the sequence is $|\pi - 3A| / 2^n$ which tends to zero as n tends to infinity. Among all triangles, the equilateral triangle thus has the largest Q-value of $\frac{3\sqrt{3}}{4} - \frac{1}{4} \approx 0.1635$.

Finally, it is worth reflecting on why the proof is a bit fiddly. First, since $f(x) = \ln\sin x$ is concave on $(0, \pi)$, $\frac{1}{3}\Sigma\ln\sin A \leqslant \ln\sin\left(\frac{1}{3}\Sigma A\right)$ or $\Pi\sin A \leqslant \sin^3\frac{\pi}{3}$ and $\frac{1}{3}\Sigma\ln\sin\frac{1}{2}A \leqslant \ln\sin\left(\frac{1}{6}\Sigma A\right)$ or $\Pi\sin^2\frac{1}{2}A \leqslant \sin^6\frac{\pi}{6}$ so the individual constituents of Q, $\frac{2}{\pi}\Pi\sin A$ and $16\Pi\sin^2\frac{1}{2}A$, are themselves separately maximised for an equilateral triangle. Second, as in the STEP question, it is easy to generate relatively tight upper bounds for Q. For example, writing $S = \Pi\sin\frac{1}{2}A$ and $C = \Pi\cos\frac{1}{2}A$ we have $Q = \frac{16}{\pi}CS - 16S^2 = \frac{16}{\pi}S(C - \pi S) \leqslant \frac{4C^2}{\pi^2}$ on maximising as a quadratic in S. But, as above, $C \leqslant \cos^3\frac{\pi}{6} = \frac{3\sqrt{3}}{8}$, so that $Q \leqslant \frac{27}{16\pi^2} \approx 0.1710$.

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91.40 An amusing sequence of trigonometrical integrals

This note was inspired by the discussion of the 8th Problem in Bailey, Borwein, Kapoor and Weisstein's beautiful recent article [1], the whole of which I warmly commend to *Gazette* readers. On pages 23-24 of this article, which is also reprised in [2], they highlight the amusing behaviour of the sequence of 'sinc integrals' of the form $\int_0^\infty \frac{\sin a_1 x}{a_1 x}\frac{\sin a_2 x}{a_2 x}\dots\frac{\sin a_n x}{a_n x}dx$ for certain choices of sequences (a_n). In this note, we consider the behaviour of the not unrelated sequence of integrals

$$\int_0^\infty \cos a_1 x \cos a_2 x \dots \cos a_n x \frac{\sin bx}{x}\, dx$$

which exhibits equally amusing behaviour. Moreover, this behaviour can be explained by a broadly similar but rather more straightforward analysis than that needed for the sequence of sinc integrals, [3].

The result we shall prove as our main theorem is not new: indeed, it may be found in the 19th century literature, as we shall remark in our final paragraph.

What, then, is the joke? Consider the following (true) statements which are fun to try to verify on a computer algebra package:

$$\int_0^\infty \cos x \frac{\sin 4x}{x}\, dx = 1.57079632679\dots ,$$

$$\int_0^\infty \cos x \cos\frac{x}{2} \frac{\sin 4x}{x}\, dx = 1.57079632679\dots ,$$

$$\int_0^\infty \cos x \cos\frac{x}{2} \cos\frac{x}{3} \frac{\sin 4x}{x}\, dx = 1.57079632679\dots ,$$

and so on, as far as

$$\int_0^\infty \cos x \cos\frac{x}{2} \cos\frac{x}{3} \cdots \cos\frac{x}{30} \frac{\sin 4x}{x}\, dx = 1.57079632\underline{679}\dots ,$$

but this is followed by

$$\int_0^\infty \cos x \cos\frac{x}{2} \cos\frac{x}{3} \cdots \cos\frac{x}{31} \frac{\sin 4x}{x}\, dx = 1.57079632\underline{533}\dots .$$

The 1.57079… is screaming $\frac{\pi}{2}$, but is the blip on the 31st term a rounding error or the start of a pattern breaking down? (Indeed, the authors of [1] were alerted to this pattern of behaviour in their sequence of integrals on hearing that, when a researcher evaluated the sequence of integrals using a well-known computer algebra package, both he and the software vendor concluded that the blip indicated a bug in the software!)

We begin our analysis with a lemma.

Lemma: Let a, b be real numbers with $b > 0$. Then:

(a) $\int_0^\infty \frac{\sin bx}{x}\, dx = \frac{\pi}{2}$,

(b) $\int_0^\infty \cos ax \frac{\sin bx}{x}\, dx = \frac{\pi}{2}$ if $|a| < b$, $\frac{\pi}{4}$ if $|a| = b$, and 0 if $|a| > b$.

Proof:

(a) Substitute $t = bx$ into the standard improper integral $\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$.

(b) $$\int_0^\infty \cos ax \frac{\sin bx}{x}\, dx = \int_0^\infty \cos|a|x \frac{\sin bx}{x}\, dx$$
$$= \frac{1}{2}\int_0^\infty \frac{\sin(b + |a|x) + \sin(b - |a|x)}{x}\, dx.$$
Using (a), the latter integral evaluates to $\frac{1}{2}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$ if $|a| < b$, to $\frac{1}{2}\left(\frac{\pi}{2} + 0\right)$ if $|a| = b$, and to $\frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{2}\right)$ if $|a| > b$.

Our main theorem is:

Theorem: Let $a_1, \ldots, a_n, b$ be positive real numbers with $a_1 + \ldots + a_n < b$. Then $\int_0^\infty \cos a_1x \cos a_2x \ldots \cos a_nx \frac{\sin bx}{x} dx = \frac{\pi}{2}$.

Proof: By repeated use of the factor formulae, we obtain some of the 'lesser-spotted' trigonometrical identities:

$$\cos a_1x \cos a_2x \equiv \tfrac{1}{2}\left[\cos(a_1 + a_2)x + \cos(a_1 - a_2)x\right]$$

$\cos a_1x \cos a_2x \cos a_3x \equiv$

$$\tfrac{1}{4}\left[\cos(a_1 + a_2 + a_3)x + \cos(a_1 + a_2 - a_3)x + \cos(a_1 - a_2 + a_3)x + \cos(a_1 - a_2 - a_3)x\right]$$

with, in general,

$$\cos a_1x \cos a_2x \ldots \cos a_nx \equiv \frac{1}{2^{n-1}} \sum_{\varepsilon_i = \pm 1} \cos(a_1 + \varepsilon_2 a_2 + \ldots + \varepsilon_n a_n)x. \qquad (1)$$

Since, for each of the 2^{n-1} choices of signs (ε_i), $|a_1 + \varepsilon_2 a_2 + \ldots + \varepsilon_n a_n| \leqslant a_1 + a_2 + \ldots + a_n < b$, the lemma ensures that

$$\int_0^\infty \cos(a_1 + \varepsilon_2 a_2 + \ldots + \varepsilon_n a_n)x \frac{\sin bx}{x} dx = \frac{\pi}{2}$$

so that, by (1),

$$\int_0^\infty \cos a_1x \cos a_2x \ldots \cos a_nx \frac{\sin bx}{x} dx = \frac{1}{2^{n-1}}\, 2^{n-1}\frac{\pi}{2} = \frac{\pi}{2},$$

as claimed.

We are now ready to explain the behaviour of the perplexing sequence highlighted above. Let (a_n) be a strictly decreasing sequence of positive real numbers and suppose that, for some $n \geqslant 2$, $a_1 + a_2 + \ldots + a_{n-1} < b < a_1 + a_2 + \ldots + a_{n-1} + a_n$. By the theorem,

$$\int_0^\infty \cos a_1x \cos a_2x \ldots \cos a_kx \frac{\sin bx}{x}\, dx = \frac{\pi}{2}$$

for all $1 \leqslant k \leqslant n - 1$.

But, in (1), $a_1 + \ldots + a_n > b$ with $-b < a_1 + \varepsilon_2 a_2 + \ldots + \varepsilon_n a_n < b$ for all other choices of signs. Thus, by the lemma applied to each of the terms in (1),

$$\int_0^\infty \cos a_1 x \cos a_2 x \ldots \cos a_n x \frac{\sin bx}{x}\, dx = \frac{1}{2^{n-1}}\left[0 + \left(2^{n-1} - 1\right)\frac{\pi}{2}\right] = \frac{\pi}{2}\left(1 - \frac{1}{2^{n-1}}\right).$$

Our opening sequence is thus fully explained by the fact that

$$\frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{30} < 4 < \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{30} + \frac{1}{31}$$

so that $\int_0^\infty \cos x \cos\frac{x}{2} \ldots \cos\frac{x}{k}\, \frac{\sin 4x}{x}\, dx = \frac{\pi}{2}$ for all $1 \leqslant k \leqslant 30$ but $\int_0^\infty \cos x \cos\frac{x}{2} \ldots \cos\frac{x}{31}\, \frac{\sin 4x}{x}\, dx = \frac{\pi}{2}\left(1 - \frac{1}{2^{30}}\right)$ – the pattern really does start breaking down! Rather agreeably, replacing $\sin 4x$ by $\sin 5x$, $\sin 6x$, …, we can make the initial sequence of $\left(\frac{1}{2}\pi\right)$s as long as we like and the ensuing breakdown as imperceptible as we like. Indeed, since $\sum_{r=1}^{n} \frac{1}{r} \approx \ln n + \gamma$, replacing $\sin 4x$ by $\sin bx$, with b a whole number, generates an initial sequence of $\left(\frac{1}{2}\pi\right)$s of length approximately $e^{b-\gamma}$ with a 'breakdown-blip' of approximately $\frac{\pi}{2}\left(1 - \frac{1}{2^{e^{b-\gamma}}}\right)$.

It is also worth noting that, because $\sum_{r=1}^{n} \frac{1}{r}$ is never an integer, [4], the '$\pi/4$-case' in part (b) of the lemma cannot arise here. And, of course, if $\sum a_n$ is a convergent series of positive real numbers with $\sum_{n=1}^{\infty} a_n < b$, then $\int_0^\infty \cos a_1 x \cos a_2 x \ldots \cos a_n x \frac{\sin bx}{x}\, dx$ is a constant sequence, being equal to $\frac{1}{2}\pi$ for all $n \geqslant 1$.

In principle, the method used in the proof of the theorem can be used to evaluate $\int_0^\infty \cos a_1 x \cos a_2 x \ldots \cos a_n x \frac{\sin bx}{x}\, dx$ for any $a_1, a_2, \ldots, a_n > 0$ (not just with $a_1 + a_2 + \ldots + a_n < b$): the sum in (1) splits into three pieces according as $\left|a_1 + \varepsilon_2 a_2 + \ldots + \varepsilon_n a_n\right| < b, = b, > b$. For example, the choice $a_1 = a_2 = \ldots = a_n = 1$ and $b = n$ yields

$$\int_0^\infty \cos^n x \frac{\sin nx}{x}\, dx = \frac{1}{2^{n-1}}\left[\frac{\pi}{4} + \left(2^{n-1} - 1\right)\frac{\pi}{2}\right] = \frac{\pi}{2}\left(1 - \frac{1}{2^n}\right).$$

(Incidentally, in this case, (1) provides an alternative to the use of de Moivre's theorem in deriving an expression for $\cos^n x$ in terms of multiple angles: counting the signs in (1) shows that

$$\cos^n x = \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} \cos(n - 2k)x$$

which equals $\frac{1}{2^{n-1}} \sum_{k=0}^{\frac{1}{2}(n-1)} \binom{n}{k} \cos(n-2k)x$ if n is odd and

$$\frac{1}{2^{n-1}} \left[\sum_{k=0}^{\frac{1}{2}n-1} \binom{n}{k} \cos(n-2k)x + \binom{n-1}{\frac{1}{2}n} \right]$$

if n is even, on pairing terms.)

Finally, it seems appropriate to end with an amusing footnote. A generalisation due to Störmer in the 19th century, which combines our main example with the sinc integrals of [1,2,3], may be found lurking in the pages of Whittaker and Watson, [5], as an exercise in the chapter on the theory of residues:

if $\phi_1, \phi_2, \ldots, \phi_n, \alpha_1, \alpha_2, \ldots, a_m, a$ are positive real numbers, then

$$\int_0^\infty \frac{\sin \phi_1 x}{\phi_1 x} \frac{\sin \phi_2 x}{\phi_2 x} \ldots \frac{\sin \phi_n x}{\phi_n x} \cos \alpha_1 x \cos \alpha_2 x \ldots \cos \alpha_m x \frac{\sin ax}{x} \, dx = \frac{\pi}{2}$$

provided $a > \phi_1 + \phi_2 + \ldots + \phi_n + \alpha_1 + \alpha_2 + \ldots + a_m$.

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91.41 Some integrals involving ln (tan*t*)

The beta function $B(p, q)$ is defined as

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \text{ for } p, q > 0$$

where $\Gamma(z)$ is the gamma function, and satisfies the identity

$$\tfrac{1}{2}B(p, q) = \int_0^{\pi/2} (\cos^{2p-1} t)(\sin^{2q-1} t)\, dt. \quad (1)$$